COUNTING THE TRAINS

TRAINS, FIBONACCI, AND RECURSIVE PATTERNS

In this lesson, students will use Cuisenaire Rods to build trains of different lengths and investigate patterns. Students will use tables to create graphs, define recursive functions, and approximate exponential formulas to describe the patterns.

LEARNING OBJECTIVES

By the end of this lesson, students will:

- Represent data using tables, graphs and rules
- Investigate patterns and make conjectures
- Explain their reasoning when making conjectures

MATERIALS

- Counting the Trains Activity Sheet
- Grid paper
- Colored pencils, markers or crayons
- Cuisenaire Rods (or paper rods in color or black and white)
- Graphing calculator
- First Five Groups in Train Pattern Overhead (optional)
- Using the TI-83 or TI-84 for Regression

INSTRUCTIONAL PLAN

To introduce the notation for recursive patterns, begin by asking students to find patterns and predict the next number in the following sequences:

1. **5, 8, 11, 14, …**
2. **4, 6, 8, 10, 12, …**

3. **10, 20, 30, 40, …**

Give students some time to work through these examples and ask them to write down how they got their answers. Have students share their answers with the class. Students may have different answers, so allow students who have different methods to explain how they got their answers. Correct any misconceptions.

Discuss students' responses with the following:

1. The next number in the first pattern is 17 (add 3 to the previous number).
   17 = 14 + 3
   Ask them how to get the next number: 20 = 17 + 3

   Students should be able to explain that to get the next number, you add 3 to the previous number.

   In mathematical language this is $\mathbf{A}_n = \mathbf{A}_{n-1} + \mathbf{3}$ and is called a *recursive rule*. *Recursion* is the root of *recursive*.

   > **Recursion** is defining the next (subsequent) term in a pattern (sequence) by using the term(s) that came before it.

   The terms *subsequent* and *sequence* are formal language that can be substituted into the definition if a formal definition is needed. If students are not used to these formal words, it may be helpful to define math terms using their vocabulary.

2. The next number in the second pattern is 14 (add 2 to the previous number): 14 = 12 + 2
   Ask them how to get the next number: 16 = 14 + 2
   Tell them that in order to get the next number, you keep adding 2 to the previous term. In mathematical language this is $\mathbf{A}_n = \mathbf{A}_{n-1} + \mathbf{2}$
3. In the third pattern, the next number is 50 (add 10 to the term before).
   Ask students to write this pattern using the new recursion notation.
   Check that students know what the subscripts refer to.

   At your discretion, you may choose to talk about the first term as the "0th" term or as the "1st" term. Note that students easily grasp that the first term is the "1st" term or $n = 1$, but may have trouble with $n = 0$ as the 1st term.

There is a rule that can be applied to all 3 of the patterns. Wait for students to think about this. If necessary, suggest *the next number in any pattern is double the term before, minus the term before that*.

Ask students to try it out for each one. Check that they see how this works.

- From pattern 1: $2 \times 14 - 11 = 17$
- From pattern 2: $2 \times 12 - 10 = 14$
- From pattern 3: $2 \times 40 - 30 = 50$

Challenge the class find the recursive rule of $\mathbf{A_n = 2A_{n-1} - A_{n-2}}$ and explain that $n - 2$ refers to the term that is 2 before.

To check for understanding, ask them to write a rule for 3, 7, 11, 15, ….

Ask students to find the next term and to write 2 recursive rules. $A_n = A_{n-1} + 4$ and $A_n = 2A_{n-1} - A_{n-2}$ will both work. If they need more examples, use either or both of the following:

- 6, 11, 16, 21, …
- 22, 25, 28, 31, …

Before moving on, make sure to explain recursion again, and make sure students have a working definition written down.

Finding Recursive Patterns Using Trains

Explain to students that they will use two lengths of cars to form trains:

Show students the trains below and explain that even though they use the same cars, they are two different trains.

a 4-train made from 2 cars of length 1 and 1 car of length 2

a 4-train train made from 1 car of length 1, 1 car of length 2, and another car of length 1

Explain that the train below is a train of length 5 made from 1 car of length 1 and 2 cars of length 2.

Give students an opportunity to ask questions, and allow them to hold the trains if it helps them to explain their thinking. Emphasizing the difference between *train length* and *car length* at the beginning helps students to talk about the different types of trains. It gives them a common language. To assess their understanding, hold up a train and ask students to describe it to you. Repeat this until all students understand. Then, have students work in groups of 2–4 to build the trains. This will decrease the number of train combinations you will have to check.

Distribute the following: train sets; Counting the Trains activity sheet; grid paper; and colored pencils, markers, or crayons. Before the lesson, prepare set of Cuisenaire train cards (or paper strips if you are using them as substitutes). Make sure that each set has 50 cars of length 1, 20 cars of length 2, 15 cars of length 3, 10 cars of length 4, and 5 cars of length 5.

Counting the Trains Activity Sheet

Students who can't see certain colors could put a number in each car. Explain that studnets should use the grid paper and the colors to record their trains. Remind students that they must build and *record* their trains. It might help to ask, "If I were to take your cars away, would you still be able to tell me what ALL the combinations you built were by reading what is on your paper?" Suggest that they use black to represent the white trains to avoid problems with recording white trains.

Make sure that students have built and recorded all of the different trains they need to complete Question 1 on the activity sheet. When students say they have them all, be sure to ask how they know they have them all. If they are missing trains, you can tell them that they are missing some. You might need to point out another train that has the same cars or point to another train of a different length that looks similar. When all else fails, give them the pieces they need and let them build the train(s) they are missing.

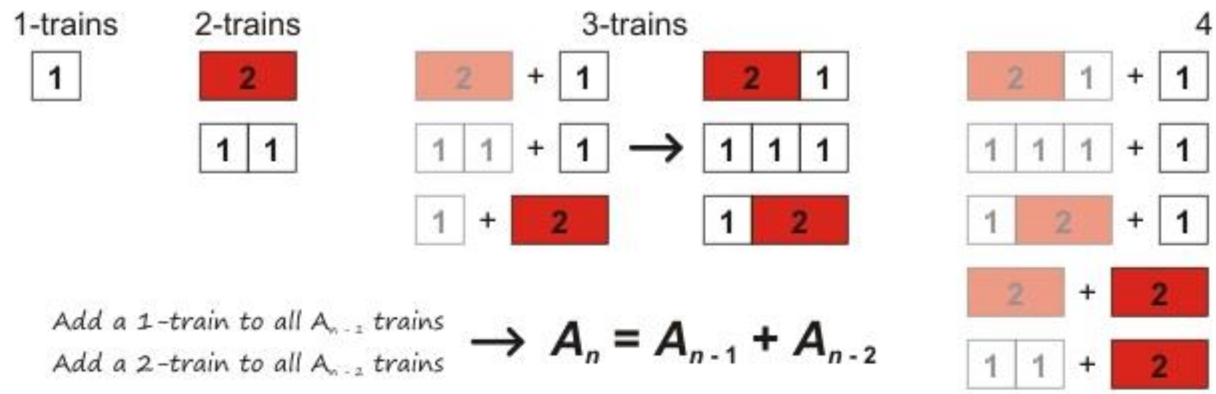
If some students finish building all the trains sooner than others, encourage them to move on to the subsequent questions. Students who complete the activity sheet before others are done can always do the *Extension* activities listed below.

Lead a whole group discussion, allowing students to present how they figured out the number of trains of length 5 and including the following questions:

- How can you get from the 3 trains of length 3 and the 5 trains of length 4 to the trains of length 5.

  [Explain or show that they can either add a 1-car to each of the trains of length 4 or add a 2-car to each of the trains of length 3.]

- Does the same method work for getting from the trains of length 4 and 5 to the trains of length 6?
- Share the following set of trains with the class.



If you have no mechanism for displaying the image above from a computer, use the blackline master provided below to create an image to use as an overhead.

First Five Groups in Train Pattern Overhead

Students should be able to see that the trains of length 5 are made from adding the trains of length 3 and length 4, and see that $A_5 = A_4 + A_3$. Ask them to explain how they made trains of length 6 and to show you the recursive rule ($A_6 = A_5 + A_4$).

At this point, ask students how to find the trains of length *n*. They should be able to tell you that you take the train before ($n - 1$) and the train before that ($n - 2$), which leads to the rule $A_n = A_{n-1} + A_{n-2}$.

Ask students to present or explain their table and scatter plot. Have students discuss any difficulties they ran into in making the table. You may want to point out that train length should be on the *x*-axis since it is the independent variable (save this discussion if this is not part of your curriculum). When discussing the scatter plot, ask students if they think the graph is linear. Have them explain their reasoning. Depending on the students' exposure to exponential functions, they may be able to explain why the function is exponential.

The common ratio for this function is approximately $(1+\sqrt{5})/2$ (which is about 1.618) and is directly related to Fibonacci numbers. A discussion of finding a regression line and a common ratio may be appropriate, depending on students' prior experience with regression and exponential functions. You may want to ask students to calculate an exponential regression for the data in Questions 4 and 5 on their activity sheet:

x	1	2	3	4	5	6	7	8	9
y	1	2	3	5	8	13	21	34	55

Directions for calculating a regression line on the TI-83/TI-84 calculator are available on the Using the TI-83 or TI-84 for Regression sheet. The calculator gives $y = 0.685 \cdot 1.632^x$ as a regression line with an *r*-squared value of 0.998, which indicates that it is a very good fit. If students use more data points, the regression comes closer and closer to

$$\frac{1}{\sqrt{5}} + \left(\frac{1+\sqrt{5}}{2}\right)^x$$

To help get at why this is, use the Golden Ratio lesson in the *Extensions* section. The Fibonacci Rabbits activity sheet from that lesson can also be used as an assessment. However, More Trains, the next lesson in this unit, involves another pattern that looks exponential and will lead students to a better understanding of lines of best fit.

QUESTIONS FOR STUDENTS

1. How do you know you have all of the trains?
2. Looking at the trains, how can you make the next length of trains from the trains that come before?
3. Does it matter which you put on the *x*-axis, train length or number of different trains?
4. In your scatter plot, should you connect the points with lines? What would the space between the points represent?
5. Can you use only the number of trains of length 9 to predict the number of trains of length 10?

ASSESSMENT OPTIONS

1. Ask students to build all the trains of length 1, 2, 3, 4 and 5 using cars of length 2, 3, 4 and 5 (no cars of length 1).
2. Have them complete the table below, write a recursive rule and make a scatter plot.

Train Length	1	2	3	4	5	6	7
Number of Ways	0						

3. [The recursive rule is the same, but the initial values are different.]

EXTENSIONS

1. Have students complete the Fibonacci Rabbits activity sheet (from the Golden Ratio lesson).
2. Have students use the Internet to research the occurrence of Fibonacci numbers in nature.
3. The More Trainslesson also involves trains, but it uses a different pattern.

TEACHER REFLECTION

- Did you challenge the achievers? How?
- What did students know already about recursion and exponential growth?
- How does this lesson relate to exponential growth?
- Did students exceed your expectations in some areas and not meet them in others?
- Were students actively engaged in the learning process? How do you know?
- Did you find it necessary to make adjustments while teaching the lesson? If so, what adjustments, and were these adjustments effective?

NCTM STANDARDS AND EXPECTATIONS

Algebra 6-8

1. Represent, analyze, and generalize a variety of patterns with tables, graphs, words, and, when possible, symbolic rules.
2. Model and solve contextualized problems using various representations, such as graphs, tables, and equations.

Algebra 9-12

1. Generalize patterns using explicitly defined and recursively defined functions.
2. Use a variety of symbolic representations, including recursive and parametric equations, for functions and relations.

This lesson prepared by Benjamin Sinwell.

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